Law of Total Tricks: Associated Probabilities

'The total number of tricks available to both sides in their longest trump suit equals the total number of cards they hold in those two fits. It is hardly a Law, more a guideline in the sense that it is sometimes out by a trick or so'.

(Andrew Robson, English Bridge, August 2017, p12-13.)

What is the Probability that you have A cards in a particular suit, then what is the Probability that your partner has X cards in that suit, giving you a suit fit of N_F = (A + X) cards?

$$\%P(A) = \%P(X) = 100 * {}^{13}C_A * {}^{39}C_{13-A} / {}^{52}C_{13}$$

$$\%P(A,X) = \%P(A) * {}^{13-A}C_X * {}^{26+A}C_{13-X} / {}^{39}C_{13}$$

Fig. 1 covers all possible suit distributions of the two hands to determine the relative probabilities of having a hand with A (or X) cards, 0 to 13, in a suit, and a fit of **F** cards, 0 to 13, in that suit.

	Figure 1	Having A cards in a suit, how many cards, X, does partner have, giving F =(A+X) for the fit in that suit?													
	X or F	0	1	2	3	4	5	6	7	8	9	10	11	12	13
	%P(F)	0.0016	0.040	0.395	2.17	7.36	16.18	23.85	23.85	16.18	7.36	2.17	0.395	0.040	0.0016
%P(A)	A														
1.28	0	0.002	0.020	0.095	0.239	0.352	0.317	0.178	0.062	0.013	0.002	0.0001	0.000004	0.000000	0.000000
8.01	1	0.020	0.206	0.848	1.829	2.287	1.733	0.809	0.231	0.039	0.004	0.0002	0.000004	0.000000	–
20.59	2	0.095	0.848	2.994	5.488	5.777	3.640	1.387	0.315	0.041	0.003	0.0001	0.000001	–	–
28.63	3	0.239	1.829	5.488	8.473	7.414	3.813	1.155	0.201	0.019	0.001	0.00001	–	–	–
23.86	4	0.352	2.287	5.777	7.414	5.296	2.166	0.502	0.063	0.004	0.0001	–	–	–	–
12.47	5	0.317	1.733	3.640	3.813	2.166	0.678	0.113	0.009	0.0003	–	–	–	–	–
4.16	6	0.178	0.809	1.387	1.155	0.502	0.113	0.012	0.0005	–	–	–	–	–	–
0.88	7	0.062	0.231	0.315	0.201	0.063	0.009	0.0005	–	–	–	–	–	–	–
0.117	8	0.013	0.039	0.041	0.019	0.004	0.0003	–	–	–	–	–	–	–	–
0.009	9	0.002	0.004	0.003	0.001	0.0001	–	–	–					–	–
0.0004	10	0.0001	0.0002	0.0001	0.00001	–	–	–	–	%Probability (A, X)				–	–
0.00001	11	0.000004	0.000004	0.000001	–	–	–	–	–					–	–
0.0000001	12	0.000000	0.000000	–	–	–	–	–	–	–	–	–	–	–	–
0.0000000	13	0.000000	–	–	–	–	–	–	–	–	–	–	–	–	–
	%P(X)	1.28	8.01	20.59	28.63	23.86	12.47	4.16	0.88	0.117	0.009	0.0004	0.00001	0.0000001	0.0000000

Selecting lines of cells, horizontal for %P(A), vertical for %P(X), or diagonal for %P(**F**) these total 100%; the built-in symmetry is evident in the table. For each side (NS or EW) there are four values of **F**, one for each suit. The highest is N_F = MAX[$\mathbf{F_{spades}}$, $\mathbf{F_{hearts}}$, $\mathbf{F_{diamonds}}$, $\mathbf{F_{clubs}}$], normally the choice for trumps although sometimes a major may be preferred because it scores more.

What is the probability of having a maximum suit fit of N_F cards? To determine this it is necessary to consider the possible suit distributions within the 26-card holding of either NS or EW whilst recognizing that these are directly linked. The number of 'Total-Fits', N_{FF}, equals N_F(NS) + N_F(EW) .

Assume NS has a maximum fit N_F of **7**; the only possible NS and EW suit splits are (**7**6**7**6 and 6**7**6**7**, 6 ways with N_{FF} = **14**) or (**777**5 and 666**8**, 4 ways with N_{FF}=**15**), the total probability being 6 x 1•75% = 10•49% for N_{FF} = **14**; 2 boards out of every 21 are expected to be perfectly balanced with neither NS or EW having a suit fit of more than 7 cards.

It turns out that 16 total-fits are only available when both NS and EW have a best suit fit, N_F , of 8 cards (**88**55 and 55**88**, 6 ways) or (**8**765 and 567**8**, 24 ways) with 6 x 0•553% plus 24 x 0•983% = 26•92%. Sorting the 1834 possible suit distributions with respect to best fit (N_F) gives Table 1 and shows that the most probable fit of 8-cards at 57%, is expected on 4 out of every 7 boards.

N_F	ways	%Probability	1 in N boards
7	10	15·74	6
8	74	57·08	2
9	198	22·13	5
10	318	4·64	22
11	394	0·35	285
12	426	0·06	1816
13	414	0·01	15793
total	1834	100·00	Table 1

Sorting the distributions with respect to total tricks (N_{FF}) gives the probabilities in Table 2, showing that 16 total fits to be available in 3 out of every 11 boards.

N_{FF}	ways	%Probability	% in 1018 boards
14	6	10·49	11.8
15	8	10·49	10.5
16	30	26·92	25.9
17	48	22·95	20.8
18	78	15·63	16.8
19	104	8·46	8.2
20	150	3·56	3.7
21	192	1·15	1.4
22	246	0·29	0.4
23	288	0·05	
24	294	0·01	
25	240	0·00	
26	150	0·00	
total	1834	100·00	Table 2

Combining these tables produces probabilities for all 14 to 26 total Best-Fits counts that can be expected given any best NS or EW suit fit of 7 to 13 cards; these are displayed as Fig. 2 with the appropriate averages.

Figure 2		Number of Total Best Fits, N_{FF}													
$P(N_F)$	NS or EW Best Suit Fit, N_F	14	15	16	17	18	19	20	21	22	23	24	25	26	Av N_{FF}
15.74%	7	66.67%	33.33%												14.3
45.74%	8		11.47%	58.86%	25.08%	4.30%	0.29%								16.2
28.10%	9				40.83%	41.64%	14.58%	2.71%	0.23%	0.01%					17.8
8.67%	10					22.68%	47.25%	23.43%	5.85%	0.75%	0.04%				19.1
1.58%	11						8.48%	48.16%	32.10%	9.67%	1.49%	0.09%			20.5
0.16%	12								41.00%	41.17%	14.94%	2.69%	0.19%		21.8
0.01%	13									24.41%	48.38%	22.17%	4.66%	0.39%	23.1

Probability (N_F, N_{FF})

Whilst it is often possible from the bidding to determine N_F for your partnership there is usually less certainty in estimating that for the opposition, and assuming N_{FF} = 2 x N_F may be the best that you can do. This is supported by the probabilities for N_F = **7** and **8** but for N_F = **9** and **10** one trick less should be considered, and for N_F =**11** two tricks less is favourite.

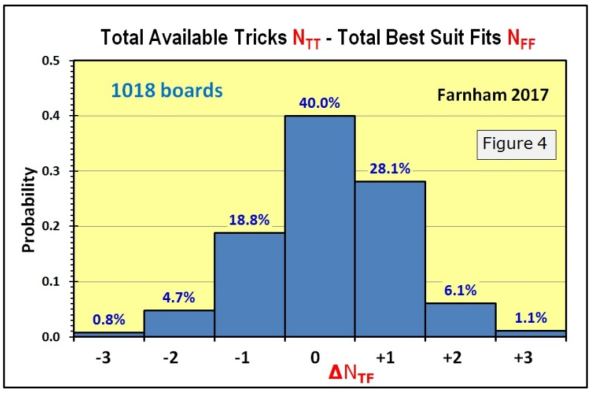
If NS has a *best-fit* of 8 then there is a 59% chance that EW also has. If NS has a *best-fit* of 9 then it is equally likely that EW has a *best-fit* of 9 (42%) or only 8 (41%).

However the value of the 'Law' rests on the implied equality of N_{FF} with the total number of available tricks N_{TT} , adding together those that NS and EW could separately win with their best suit as trumps.

Duplimate computer dealt boards with ScoreBridge records provide the data required to test the validity of the 'Law'. 34 sets of boards, 1018 in total, produced by Farnham Bridge Club were selected for analysis of the total *best-fits* N_{FF}, ranging between 14 and 22, when those for NS and EW are added together for each board; the relevant percentages are added to Table 2, matching the expected probabilities fairly well.

$\Delta N_{TF} = N_{TT} - N_{FF}$	Total Best Fits N_{FF} 14	15	16	17	18	19	20	21	22	total	Probability
3	2	0	4	1	2	1	1	0	0	11	1.1%
2	8	7	15	14	13	4	1	0	0	62	6.1%
1	52	27	78	60	49	16	4	0	0	286	28.1%
0	54	60	105	83	58	32	11	3	1	407	40.0%
-1	4	12	58	45	34	27	5	6	0	191	18.8%
-2	0	1	4	6	14	3	15	4	1	48	4.7%
-3	0	0	0	3	1	0	1	1	2	8	0.8%
	120	107	264	212	171	83	38	14	4	1018	99.51%
Prob	11.8%	10.5%	25.9%	20.8%	16.8%	8.2%	3.7%	1.4%	0.4%	Figure 3	
AV Δ	0.6	0.3	0.2	0.1	0.1	-0.1	-0.8	-1.2	-2.0		

The corresponding total available tricks N_{TT} on each board were extracted and the differences ΔN_{TF} = (N_{TT} – N_{FF}) calculated. Those between -3 and +3, for each associated N_{FF} from 14 to 22, 1013 in all, were counted, listed on Fig. 3, and plotted as Fig. 4.



The peak 40.0% with zero difference confirms that the Law has credence, but the 28.1% yielding +1 and the 18.8% with -1 prove that this is somewhat limited.

Does the difference ΔN_{TF} depend upon N_{FF}? A slight downward trend with increasing fit length is evident in Fig. 3 but the average only drops from about +0•5 to -0•5.

The fact that only 5 out of 1018 boards fell outside the selected limits is comforting but a close look at one of these, shown here, should raise a note of caution.

Bd: 9 Vul: E/W Dlr: North	♠ 8762 ♥ AQJ ♦ T5432 ♣ 6	
♠ QT954 ♥ 943 ♦ K976 ♣ J	N W 9 E S	♠ AK53 ♥ 72 ♦ AJ8 ♣ A543
7 6 HCP 17 10	♠ — ♥ KT865 ♦ Q ♣KQT9872	♣♦♥♠N N 5 - 5 - - S 5 - 5 - - E - 3 - 5 2 W - 3 - 5 2

N_{FF} = 8 + 9 = 17, N_{TT} = 11 + 11 = 22, ΔN_{TF} = 22 – 17 = **5**
Both partnerships might easily calculate their fit lengths and count likely losers but it is tough to reckon that 22 total tricks may be available. In practice three Souths, having only 4 losers, pushed on to the 5-level but no EW pairs ventured above 4S.

Board No 9 E/W Vul Dealer North										
Pairs		Contract				Scores		Points		
N/S	E/W	Bid	By	Ld	Tks	N/S	E/W	N/S	E/W	
2	11	♣ 5*	S	♠ 10	10		100	6.9	9.1	
3	13	♠ 3	W	♣ 6	10		170	3.5	12.5	
6	18	♠ 4	E	♣ K	10		620	0.1	15.9	
7	20	♥ 3	S	♠ 10	8		50	9.1	6.9	
8	12	♥ 5	S	♣ J	11	450		13.6	2.4	
9	15	♠ 3	E	♣ K	10		170	3.5	12.5	
10	17	♠ 3	W	♣ 6	8	100		11.4	4.6	
16	5	♣ 5*	S	♠ 4	11	550		15.9	0.1	

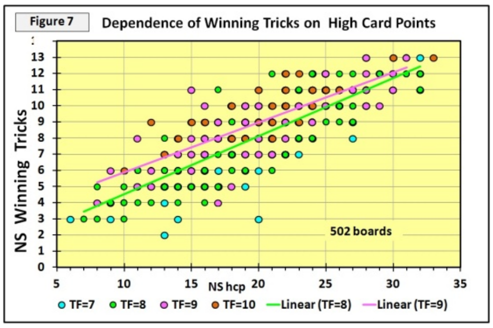
Robson unsurprisingly focuses on the battle of the majors. 189 suit distributions form a sub-set with NS and EW having a *best-fit* in hearts or spades, the probability of this being 14.37%.
When you have a 7 to 13 card fit in a major, the probability that the opposition have a fit in the other is given by Fig. 5 together with the break down in terms of the number of cards.

		EW best suit fit , other Major							
	N_F	7	8	9	10	11	12	13	Probability
NS best suit fit, Major	7	72.73%	27.27%						4.81%
	8	23.59%	45.34%	24.57%	5.90%	0.60%			5.56%
	9		47.35%	36.54%	13.15%	2.68%	0.27%	0.01%	2.88%
	10		35.33%	40.89%	18.47%	4.66%	0.61%	0.03%	0.93%
	11		19.85%	45.75%	25.60%	7.55%	1.17%	0.07%	0.17%
	12			49.37%	36.55%	12.65%	1.27%	0.16%	0.02%
	13	**Figure 5**		37.49%	41.49%	17.15%	3.58%	0.30%	0.00%
								total	14.37%

'The most common scenario is both sides having an 8-card major suit fit. You will bid to 2♥ and they will naturally bid on to 2♠'. Whilst you can know that you have an *8-card fit*, it is most likely that their bid shows an *8-card fit* (N_{FF} =**16**, P=**45%**) but they could have 7 (N_{FF} =15, P=24%) or 9 (N_{FF} =17, P=25%). Assuming they do have 8, N_{FF} =16**,** then Fig. 3 empirically gives probabilities of 19% for N_{TT} = 15, 40% for N_{TT} = 16, and 28% for N_{TT} = 17.
Combining the two sets of probabilities and adding the relevant elements produces Fig. 6 for the spade-heart fits that Robson considers.

		Total Best Fits, N_{FF}		
	♠-♥	9-9	9-8	8-8
Total Tricks, N_{TT}	13	**Figure 6**		0%
	14		0%	3%
	15	1%	1%	23%
	16	13%	13%	29%
	17	26%	26%	25%
	18	30%	26%	10%
	19	19%	14%	2%
	20	5%	3%	
	21	1%		
	totals	95%	82%	92%

'If there is an even number of trumps (and tricks), hearts will declare; if there is an odd number, spades will declare.'
For 8-8 and 9-9 ♠-♥ fits, the probabilities support the Law in slightly favouring 16 and 18 total tricks respectively. However, for 9-8 they suggest that 18 rather than 17 may well be available so 4♥ over 3♠ could often be worth the gamble.
The decision on whether to risk a sacrifice can turn on red or green vulnerability.
Understanding the Law of Total Tricks was never easy but stirring in the relevant probabilities confirms that it needs to be applied with fingers crossed.



Of course *total high card points* cannot be ignored – Fig. 7 plots available winning tricks against total ***hcp*** for NS on 502 boards, colour coded with respect to total *best-fit* (TF) with trend-lines to show the obvious dependency. With only one exception, the partnership with more than 20 hcp could win a majority of tricks (although their opponents may be able to win more).

glw, 21-11-2017